Numerical tables of finite multiple zeta values (Various aspects of multiple zeta values)

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Numerical tables of finite multiple zeta values

By

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Abstract

We review various relations among finite multiple zeta(-star) values and provide numerical tables of the values of weight at most 8.

§ 1. Finite multiple zeta values

An *index* is a finite (possibly empty) sequence of positive integers. For an index $\boldsymbol{k} = (k_1, \dots, k_n)$ and a prime p, we consider modulo p multiple harmonic sums

$$\zeta_p(\boldsymbol{k}) = \sum_{p > m_1 > \dots > m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}} \in \mathbb{Z}/p\mathbb{Z},$$
$$\zeta_p^\star(\boldsymbol{k}) = \sum_{p > m_1 \geq \dots \geq m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}} \in \mathbb{Z}/p\mathbb{Z};$$

we then look at the sums collectively for all p in the $\mathbb{Q}$-algebra

$$\mathcal{A} = \left(\prod_p (\mathbb{Z}/p\mathbb{Z}) \right) \Big/ \left(\bigoplus_p (\mathbb{Z}/p\mathbb{Z}) \right) \cong \left(\prod_p (\mathbb{Z}/p\mathbb{Z}) \right) \underset{\mathbb{Z}}{\otimes} \mathbb{Q},$$

following the idea of Kaneko and Zagier [3]:

Definition 1.1. For an index $\boldsymbol{k}$, we define the *finite multiple zeta value* (FMZV) $\zeta_{\mathcal{A}}(\boldsymbol{k})$ and the *finite multiple zeta-star value* (FMZSV) $\zeta_{\mathcal{A}}^\star(\boldsymbol{k})$ by

$$\zeta_{\mathcal{A}}(\boldsymbol{k}) = (\zeta_p(\boldsymbol{k}))_p \in \mathcal{A}, \qquad \zeta_{\mathcal{A}}^\star(\boldsymbol{k}) = (\zeta_p^\star(\boldsymbol{k}))_p \in \mathcal{A}.$$

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Table 1. Conjectured dimensions of the vector spaces $\mathcal{Z}_{\mathcal{A},k}$.

k	0	1	2	3	4	5	6	7	8	9
$\dim \mathcal{Z}_{\mathcal{A},k}$	1	0	0	1	0	1	1	1	2	2

We say that each index $\boldsymbol{k} = (k_1, \dots, k_n)$ has *weight* $\operatorname{wt} \boldsymbol{k} = k_1 + \cdots + k_n$; the empty index is viewed as having weight 0. For $k \in \mathbb{Z}_{\geq 0}$, the dimension of the $\mathbb{Q}$-vector subspace

$$\begin{aligned}\mathcal{Z}_{\mathcal{A},k} &= \operatorname{span}\{\zeta_{\mathcal{A}}(\boldsymbol{k}) \mid \boldsymbol{k} \text{ is an index of weight } k\} \\ &= \operatorname{span}\{\zeta^\star_{\mathcal{A}}(\boldsymbol{k}) \mid \boldsymbol{k} \text{ is an index of weight } k\} \subset \mathcal{A}\end{aligned}$$

is conjectured to be given by the following formula (Table 1):

Conjecture 1 (Zagier [3]). If $(d_k)_{k \in \mathbb{Z}_{\geq 0}}$ is the sequence given by $d_0 = 1$, $d_1 = d_2 = 0$, and $d_k = d_{k-2} + d_{k-3}$ for $k \in \mathbb{Z}_{\geq 3}$, then $\dim \mathcal{Z}_{\mathcal{A},k} = d_k$.

§ 2. Relations among finite multiple zeta(-star) values

This section describes various relations among finite multiple zeta(-star) values. Many of them have already been given in [1, 2, 4, 5] in equivalent forms, though some statements and some proofs have not been given explicitly in the literature.

§ 2.1. Zeta and zeta-star values

As in the case of ordinary multiple zeta(-star) values in $\mathbb{R}$, each FMZSV can be written as a $\mathbb{Z}$-linear combination of FMZVs of the same weight and vice versa, as illustrated by

$$\begin{aligned}\zeta^\star_{\mathcal{A}}(k_1, k_2, k_3) &= \zeta_{\mathcal{A}}(k_1, k_2, k_3) + \zeta_{\mathcal{A}}(k_1 + k_2, k_3) + \zeta_{\mathcal{A}}(k_1, k_2 + k_3) + \zeta_{\mathcal{A}}(k_1 + k_2 + k_3), \\ \zeta_{\mathcal{A}}(k_1, k_2, k_3) &= \zeta^\star_{\mathcal{A}}(k_1, k_2, k_3) - \zeta^\star_{\mathcal{A}}(k_1 + k_2, k_3) - \zeta^\star_{\mathcal{A}}(k_1, k_2 + k_3) + \zeta^\star_{\mathcal{A}}(k_1 + k_2 + k_3).\end{aligned}$$

§ 2.2. Single zeta values

Lemma 2.1. *If k is an integer and p is a prime, then in $\mathbb{Z}/p\mathbb{Z}$ we have*

$$\sum_{m=1}^{p-1} m^k = \begin{cases} -1 & \text{if } p-1 \text{ divides } k; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $p-1$ divides k, then Fermat's little theorem implies that $m^k = 1$ in $\mathbb{Z}/p\mathbb{Z}$ for all $m = 1, \ldots, p-1$, so that

$$\sum_{m=1}^{p-1} m^k = \sum_{m=1}^{p-1} 1 = p-1 = -1$$

in $\mathbb{Z}/p\mathbb{Z}$. Otherwise, taking a primitive root a modulo p, we have $a^k \neq 1$ in $\mathbb{Z}/p\mathbb{Z}$ and so

$$\sum_{m=1}^{p-1} m^k = \sum_{j=0}^{p-2} a^{jk} = \frac{a^{k(p-1)}-1}{a^k-1} = 0$$

in $\mathbb{Z}/p\mathbb{Z}$. □

We say that each index $\boldsymbol{k} = (k_1, \ldots, k_n)$ has *depth* $\operatorname{dep}\boldsymbol{k} = n$; the empty index is viewed as having depth 0.

Proposition 2.2. *If $\boldsymbol{k} = (k)$ is an index of depth 1, then $\zeta_{\mathcal{A}}(\boldsymbol{k}) = \zeta_{\mathcal{A}}^{\star}(\boldsymbol{k}) = 0$.*

Proof. If p is a prime with $p > k+1$, then $p-1$ cannot divide k, and so Lemma 2.1 shows that $\sum_{m=1}^{p-1} m^{-k} = 0$ in $\mathbb{Z}/p\mathbb{Z}$. We therefore have $\zeta_{\mathcal{A}}(k) = \zeta_{\mathcal{A}}^{\star}(k) = 0$. □

§ 2.3. Double zeta values and Bernoulli numbers

The *Bernoulli numbers* B_n, where $n \in \mathbb{Z}_{\geq 0}$, are defined by their generating function

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{xe^x}{e^x-1}.$$

The first few terms are $B_0 = 1$, $B_1 = 1/2$, and $B_2 = 1/6$; we have $B_n = 0$ if n is odd and $n \geq 3$, because $xe^x/(e^x-1) - x/2$ is an even function.

If $a_p \in \mathbb{Z}/p\mathbb{Z}$ for each prime p, then we write $(a_p)_p \in \mathcal{A}$ simply as a_p. Since finitely many components do not matter in $\mathcal{A}$, it makes sense to consider such an element of $\mathcal{A}$ even when a_p is not defined for finitely many p; for example B_{p-3} makes perfect sense as an element of $\mathcal{A}$ despite being undefined for $p = 2$.

Proposition 2.3 ([1, 5])**.** *If $\boldsymbol{k} = (k_1, k_2)$ is an index of depth 2, then*

$$\zeta_{\mathcal{A}}(\boldsymbol{k}) = \zeta_{\mathcal{A}}^{\star}(\boldsymbol{k}) = \frac{(-1)^{k_1}}{k_1+k_2}\binom{k_1+k_2}{k_1} B_{p-k_1-k_2}.$$

In particular, we have $\zeta_{\mathcal{A}}(k-1,1) = \zeta_{\mathcal{A}}^{\star}(k-1,1) = B_{p-k}$ for all $k \in \mathbb{Z}_{\geq 2}$.

Proof. Since $\zeta^\star_{\mathcal{A}}(\boldsymbol{k}) = \zeta_{\mathcal{A}}(\boldsymbol{k}) + \zeta_{\mathcal{A}}(k_1+k_2)$, we have $\zeta_{\mathcal{A}}(\boldsymbol{k}) = \zeta^\star_{\mathcal{A}}(\boldsymbol{k})$ by Proposition 2.2.

If p is a prime with $p > k_1+k_2+1$, then Fermat's little theorem and Faulhaber's formula show that

$$\begin{aligned}
\zeta^\star_p(k_1,k_2) &= \sum_{p>m_1\geq m_2\geq 1} \frac{1}{m_1^{k_1} m_2^{k_2}} = \sum_{m_1=1}^{p-1} \frac{1}{m_1^{k_1}} \sum_{m_2=1}^{m_1} \frac{1}{m_2^{k_2}} \\
&= \sum_{m_1=1}^{p-1} \frac{1}{m_1^{k_1}} \sum_{m_2=1}^{m_1} m_2^{p-1-k_2} = \frac{1}{p-k_2} \sum_{m_1=1}^{p-1} \frac{1}{m_1^{k_1}} \sum_{j=0}^{p-1-k_2} \binom{p-k_2}{j} B_j m_1^{p-k_2-j} \\
&= \frac{1}{p-k_2} \sum_{j=0}^{p-1-k_2} \binom{p-k_2}{j} B_j \sum_{m_1=1}^{p-1} m_1^{p-k_1-k_2-j}
\end{aligned}$$

in $\mathbb{Z}/p\mathbb{Z}$. Since $-p+1 < -k_1+1 \leq p-k_1-k_2-j \leq p-k_1-k_2 < p-1$ for $j=0,\dots,p-1-k_2$, Lemma 2.1 shows that

$$\begin{aligned}
\zeta^\star_p(k_1,k_2) &= -\frac{1}{p-k_2}\binom{p-k_2}{p-k_1-k_2} B_{p-k_1-k_2} \\
&= -\frac{(p-k_2-1)\cdots(p-k_1-k_2+1)}{k_1!} B_{p-k_1-k_2} \\
&= \frac{(-1)^{k_1}(k_2+1)\cdots(k_1+k_2-1)}{k_1!} B_{p-k_1-k_2} \\
&= \frac{(-1)^{k_1}}{k_1+k_2}\binom{k_1+k_2}{k_1} B_{p-k_1-k_2}
\end{aligned}$$

in $\mathbb{Z}/p\mathbb{Z}$, from which the proposition follows. □

Corollary 2.4. *If $\boldsymbol{k} = (k_1,k_2)$ is an index of even weight and depth 2, then $\zeta_{\mathcal{A}}(\boldsymbol{k}) = \zeta^\star_{\mathcal{A}}(\boldsymbol{k}) = 0$.*

Proof. Proposition 2.3 implies the corollary because for all sufficiently large primes p, the number $p-k_1-k_2$ is an odd integer at least 3, for which the Bernoulli number is 0. □

Corollary 2.5. *If $\boldsymbol{k} = (k_1,k_2,k_3)$ is an index of even weight and depth 3, then $\zeta_{\mathcal{A}}(\boldsymbol{k}) = \zeta^\star_{\mathcal{A}}(\boldsymbol{k})$.*

Proof. We have

$$\zeta^\star_{\mathcal{A}}(\boldsymbol{k}) - \zeta_{\mathcal{A}}(\boldsymbol{k}) = \zeta_{\mathcal{A}}(k_1+k_2,k_3) + \zeta_{\mathcal{A}}(k_1,k_2+k_3) + \zeta_{\mathcal{A}}(k_1+k_2+k_3) = 0$$

by Proposition 2.2 and Corollary 2.4. □

§ 2.4. Reversal

Proposition 2.6 ([1, 5]). *If $\boldsymbol{k} = (k_1, \ldots, k_n)$ is an index and $\boldsymbol{k}' = (k_n, \ldots, k_1)$ is its reversal, then $\zeta_{\mathcal{A}}(\boldsymbol{k}') = (-1)^{\mathrm{wt}\,\boldsymbol{k}}\zeta_{\mathcal{A}}(\boldsymbol{k})$ and $\zeta^{\star}_{\mathcal{A}}(\boldsymbol{k}') = (-1)^{\mathrm{wt}\,\boldsymbol{k}}\zeta^{\star}_{\mathcal{A}}(\boldsymbol{k})$.*

Proof. For each prime p, we have

$$\begin{aligned}\zeta_p(\boldsymbol{k}') &= \sum_{p>m_1>\cdots>m_n\geq 1} \frac{1}{m_1^{k_n}\cdots m_n^{k_1}} \\ &= (-1)^{\mathrm{wt}\,\boldsymbol{k}} \sum_{p>p-m_n>\cdots>p-m_1\geq 1} \frac{1}{(p-m_1)^{k_n}\cdots(p-m_n)^{k_1}} \\ &= (-1)^{\mathrm{wt}\,\boldsymbol{k}}\zeta_p(\boldsymbol{k})\end{aligned}$$

in $\mathbb{Z}/p\mathbb{Z}$, from which it follows that $\zeta_{\mathcal{A}}(\boldsymbol{k}') = (-1)^{\mathrm{wt}\,\boldsymbol{k}}\zeta_{\mathcal{A}}(\boldsymbol{k})$. We may prove $\zeta^{\star}_{\mathcal{A}}(\boldsymbol{k}') = (-1)^{\mathrm{wt}\,\boldsymbol{k}}\zeta^{\star}_{\mathcal{A}}(\boldsymbol{k})$ in the same manner. □

§ 2.5. Harmonic product

As in the case of ordinary multiple zeta(-star) values in $\mathbb{R}$, FMZ(S)Vs obey the *harmonic* (also known as *stuffle*) *product* rule, as illustrated by

$$\begin{aligned}\zeta_{\mathcal{A}}(k_1)\zeta(k_2,k_3) &= \zeta_{\mathcal{A}}(k_1,k_2,k_3) + \zeta_{\mathcal{A}}(k_2,k_1,k_3) + \zeta_{\mathcal{A}}(k_2,k_3,k_1) \\ &\qquad + \zeta_{\mathcal{A}}(k_1+k_2,k_3) + \zeta_{\mathcal{A}}(k_2,k_1+k_3), \\ \zeta^{\star}_{\mathcal{A}}(k_1)\zeta^{\star}(k_2,k_3) &= \zeta^{\star}_{\mathcal{A}}(k_1,k_2,k_3) + \zeta^{\star}_{\mathcal{A}}(k_2,k_1,k_3) + \zeta^{\star}_{\mathcal{A}}(k_2,k_3,k_1) \\ &\qquad - \zeta^{\star}_{\mathcal{A}}(k_1+k_2,k_3) - \zeta^{\star}_{\mathcal{A}}(k_2,k_1+k_3).\end{aligned}$$

Proposition 2.7 ([1]). *If $\boldsymbol{k} = (k_1, \ldots, k_n)$ is a nonempty index, then*

$$\sum_{\sigma\in\mathfrak{S}_n} \zeta_{\mathcal{A}}(k_{\sigma(1)},\ldots,k_{\sigma(n)}) = \sum_{\sigma\in\mathfrak{S}_n} \zeta^{\star}_{\mathcal{A}}(k_{\sigma(1)},\ldots,k_{\sigma(n)}) = 0,$$

where $\mathfrak{S}_n$ denotes the symmetric group of order n.

Proof. It suffices to prove that the sum for $\zeta_{\mathcal{A}}$, written as $S(\boldsymbol{k})$, always vanishes. We proceed by induction on n. If $n = 1$, then the statement immediately follows from Proposition 2.2. If $n = 2$, then Proposition 2.2 and the harmonic product rule show that

$$0 = \zeta_{\mathcal{A}}(k_1)\zeta_{\mathcal{A}}(k_2) = S(\boldsymbol{k}) + S(k_1+k_2)$$

and so we have $S(\boldsymbol{k}) = 0$ by the inductive hypothesis. If $n = 3$, we similarly have

$$\begin{aligned}0 &= \zeta_{\mathcal{A}}(k_1)\zeta_{\mathcal{A}}(k_2)\zeta_{\mathcal{A}}(k_3) \\ &= S(\boldsymbol{k}) + S(k_1+k_2,k_3) + S(k_1+k_3,k_2) + S(k_2+k_3,k_1) + S(k_1+k_2+k_3),\end{aligned}$$

and so we have $S(\boldsymbol{k})=0$ by the inductive hypothesis. The same method works for any n. $\square$

Corollary 2.8 ([5] for the $\zeta_{\mathcal{A}}$ case)**.** *If $\boldsymbol{k}=(k,\dots,k)$ is a nonempty index whose components are all equal, then $\zeta_{\mathcal{A}}(\boldsymbol{k})=\zeta^{\star}_{\mathcal{A}}(\boldsymbol{k})=0$.*

Proof. If n is the depth of $\boldsymbol{k}$, then Proposition 2.7 shows that $n!\zeta_{\mathcal{A}}(\boldsymbol{k})=n!\zeta^{\star}_{\mathcal{A}}(\boldsymbol{k})=0$, from which the corollary follows. $\square$

Proposition 2.9 (e.g. [2])**.** *If $\boldsymbol{k}=(k_1,\dots,k_n)$ is a nonempty index, then*

$$\sum_{j=0}^{n}(-1)^j\zeta^{\star}_{\mathcal{A}}(k_j,\dots,k_1)\zeta_{\mathcal{A}}(k_{j+1},\dots,k_n)=0.$$

Proof. Set $S_j=\zeta^{\star}_{\mathcal{A}}(k_j,\dots,k_1)\zeta_{\mathcal{A}}(k_{j+1},\dots,k_n)$ for $j=0,\dots,n$. Expanding $\zeta^{\star}_{\mathcal{A}}(k_j,\dots,k_1)$ in terms of FMZVs and using the harmonic product rule, we can write S_j as a sum of FMZVs. For $j=1,\dots,n-1$, we write $S_j=T_j+U_j$ by letting T_j be the sum of those FMZVs in which the component containing k_j appears strictly left of that containing k_{j+1}, and by letting U_j be the sum of the other FMZVs. If we further set $U_0=S_0$ and $T_n=S_n$, then we may observe that $U_j=T_{j+1}$ for $j=0,\dots,n-1$; for example, if $n=3$, then

$$\begin{aligned}
U_0&=T_1=\zeta_{\mathcal{A}}(k_1,k_2,k_3),\\
U_1&=T_2=\zeta_{\mathcal{A}}(k_2,k_1,k_3)+\zeta_{\mathcal{A}}(k_2,k_3,k_1)+\zeta_{\mathcal{A}}(k_1+k_2,k_3)+\zeta_{\mathcal{A}}(k_2,k_1+k_3),\\
U_2&=T_3=\zeta_{\mathcal{A}}(k_3,k_2,k_1)+\zeta_{\mathcal{A}}(k_2+k_3,k_1)+\zeta_{\mathcal{A}}(k_3,k_1+k_2)+\zeta_{\mathcal{A}}(k_1+k_2+k_3).
\end{aligned}$$

It follows that

$$\sum_{j=0}^{n}(-1)^jS_j=\sum_{j=0}^{n-1}(-1)^j(U_j-T_{j+1})=0,$$

as desired. $\square$

Corollary 2.10 ([1])**.** *If $\boldsymbol{k}=(k_1,k_2,k_3)$ is an index of odd weight k and depth 3, then*

$$\begin{aligned}
\zeta_{\mathcal{A}}(\boldsymbol{k})&=\left(-(-1)^{k_1}\binom{k}{k_1}+(-1)^{k_3}\binom{k}{k_3}\right)\frac{B_{p-k}}{2k},\\
\zeta^{\star}_{\mathcal{A}}(\boldsymbol{k})&=\left(\quad(-1)^{k_1}\binom{k}{k_1}-(-1)^{k_3}\binom{k}{k_3}\right)\frac{B_{p-k}}{2k}.
\end{aligned}$$

Proof. Propositions 2.2, 2.6 and 2.9 give

$$\begin{aligned}
0&=\zeta_{\mathcal{A}}(k_1,k_2,k_3)-\zeta^{\star}_{\mathcal{A}}(k_1)\zeta_{\mathcal{A}}(k_2,k_3)+\zeta^{\star}_{\mathcal{A}}(k_2,k_1)\zeta_{\mathcal{A}}(k_3)-\zeta^{\star}_{\mathcal{A}}(k_3,k_2,k_1)\\
&=\zeta_{\mathcal{A}}(\boldsymbol{k})+\zeta^{\star}_{\mathcal{A}}(\boldsymbol{k}).
\end{aligned}$$

Also, Propositions 2.2 and 2.3 show that

$$\begin{aligned}\zeta^\star_{\mathcal{A}}(\boldsymbol{k}) &= \zeta_{\mathcal{A}}(\boldsymbol{k}) + \zeta_{\mathcal{A}}(k_1+k_2,k_3) + \zeta_{\mathcal{A}}(k_1,k_2+k_3) + \zeta_{\mathcal{A}}(k_1+k_2+k_3)\\ &= \zeta_{\mathcal{A}}(\boldsymbol{k}) + \left(\frac{(-1)^{k_1+k_2}}{k}\binom{k}{k_1+k_2} + \frac{(-1)^{k_1}}{k}\binom{k}{k_1}\right)B_{p-k}\\ &= \zeta_{\mathcal{A}}(\boldsymbol{k}) + \left((-1)^{k_1}\binom{k}{k_1} - (-1)^{k_3}\binom{k}{k_3}\right)\frac{B_{p-k}}{k}\end{aligned}$$

because $k = k_1+k_2+k_3$ is odd, from which the conclusion follows. □

Corollary 2.11. *Let $\boldsymbol{k} = (k_1,k_2,k_3,k_4)$ be an index of depth 4.*

1. *If $\boldsymbol{k}$ has odd weight, then $\zeta_{\mathcal{A}}(\boldsymbol{k}) = \zeta^\star_{\mathcal{A}}(\boldsymbol{k})$.*
2. *If $\boldsymbol{k}$ has even weight, then $\zeta_{\mathcal{A}}(\boldsymbol{k}) + \zeta^\star_{\mathcal{A}}(\boldsymbol{k}) = \zeta_{\mathcal{A}}(k_1,k_2)\zeta_{\mathcal{A}}(k_3,k_4)$. In particular, $\zeta_{\mathcal{A}}(\boldsymbol{k}) + \zeta^\star_{\mathcal{A}}(\boldsymbol{k}) = 0$ unless k_1+k_2 and k_3+k_4 are both odd.*

Proof. Propositions 2.2, 2.6 and 2.9 give

$$\begin{aligned}0 &= \zeta_{\mathcal{A}}(k_1,k_2,k_3,k_4) - \zeta^\star_{\mathcal{A}}(k_1)\zeta_{\mathcal{A}}(k_2,k_3,k_4) + \zeta^\star_{\mathcal{A}}(k_2,k_1)\zeta_{\mathcal{A}}(k_3,k_4)\\ &\quad - \zeta^\star_{\mathcal{A}}(k_3,k_2,k_1)\zeta_{\mathcal{A}}(k_4) + \zeta^\star_{\mathcal{A}}(k_4,k_3,k_2,k_1)\\ &= \zeta_{\mathcal{A}}(\boldsymbol{k}) + \zeta_{\mathcal{A}}(k_2,k_1)\zeta_{\mathcal{A}}(k_3,k_4) + (-1)^{\operatorname{wt}\boldsymbol{k}}\zeta^\star_{\mathcal{A}}(\boldsymbol{k}).\end{aligned}$$

If $\operatorname{wt}\boldsymbol{k}$ is even, this gives the desired result because $\zeta_{\mathcal{A}}(k_2,k_1) = -\zeta_{\mathcal{A}}(k_1,k_2)$ no matter whether k_1+k_2 is odd or even; if $\operatorname{wt}\boldsymbol{k}$ is odd, then the middle term vanishes because either k_1+k_2 or k_3+k_4 must be even. □

§ 2.6. Duality

Definition 2.12 ([1]). For each nonempty index $\boldsymbol{k}$, we define its *dual* index $\boldsymbol{k}^*$ inductively as follows:

1. $(1)^* = (1)$;
2. if $(k_1,\ldots,k_m)^* = (l_1,\ldots,l_n)$, then

$$(1,k_1,\ldots,k_m)^* = (l_1+1,l_2,\ldots,l_n), \qquad (k_1+1,k_2,\ldots,k_m)^* = (1,l_1,\ldots,l_n).$$

Example 2.13. We have $(1,2,2)^* = (2,2,1)$ because successive use of item 2 gives $(2)^* = (1,1)$, $(1,2)^* = (2,1)$, $(2,2)^* = (1,2,1)$, and $(1,2,2)^* = (2,2,1)$, as desired.

Note that induction on weight shows that $\operatorname{wt}\boldsymbol{k}^* = \operatorname{wt}\boldsymbol{k}$ and $\operatorname{dep}\boldsymbol{k}^* = \operatorname{wt}\boldsymbol{k} - \operatorname{dep}\boldsymbol{k} + 1$ for all nonempty indices $\boldsymbol{k}$.

Theorem 2.14 ([1, Theorem 4.6]). *If $\boldsymbol{k}$ is a nonempty index, then we have*

$$\zeta_{\mathcal{A}}^{\star}(\boldsymbol{k}^*) = -\zeta_{\mathcal{A}}^{\star}(\boldsymbol{k}).$$

For indices $\boldsymbol{k}$ and $\boldsymbol{l}$ of the same weight, we write $\boldsymbol{k} \preceq \boldsymbol{l}$ if $\boldsymbol{l}$ is a refinement of $\boldsymbol{k}$, namely if $\zeta_{\mathcal{A}}(\boldsymbol{k})$ appears in the expansion of $\zeta_{\mathcal{A}}^{\star}(\boldsymbol{l})$ in terms of FMZVs.

Corollary 2.15. *If $\boldsymbol{k}$ is a nonempty index, then*

$$(-1)^{\operatorname{dep}\boldsymbol{k}}\zeta_{\mathcal{A}}(\boldsymbol{k}) = \sum_{\boldsymbol{l}\succeq\boldsymbol{k}} \zeta_{\mathcal{A}}(\boldsymbol{l}).$$

Proof. Observe that for indices $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ of the same weight, we have $\boldsymbol{k}_1 \preceq \boldsymbol{k}_2$ if and only if $\boldsymbol{k}_1^* \succeq \boldsymbol{k}_2^*$. If $\boldsymbol{k}$ is a nonempty index of weight k and depth n, then Theorem 2.14 shows that

$$\begin{aligned}
(-1)^n\zeta_{\mathcal{A}}(\boldsymbol{k}) &= \sum_{\boldsymbol{m}\preceq\boldsymbol{k}} (-1)^{\operatorname{dep}\boldsymbol{m}}\zeta_{\mathcal{A}}^{\star}(\boldsymbol{m}) = \sum_{\boldsymbol{m}\preceq\boldsymbol{k}} (-1)^{\operatorname{dep}\boldsymbol{m}+1}\zeta_{\mathcal{A}}^{\star}(\boldsymbol{m}^*) \\
&= \sum_{\boldsymbol{m}^*\succeq\boldsymbol{k}^*} (-1)^{k+\operatorname{dep}\boldsymbol{m}^*}\zeta_{\mathcal{A}}^{\star}(\boldsymbol{m}^*) = (-1)^k \sum_{\boldsymbol{m}\succeq\boldsymbol{k}^*} (-1)^{\operatorname{dep}\boldsymbol{m}}\zeta_{\mathcal{A}}^{\star}(\boldsymbol{m}) \\
&= (-1)^k \sum_{\boldsymbol{m}\succeq\boldsymbol{k}^*} (-1)^{\operatorname{dep}\boldsymbol{m}} \sum_{\boldsymbol{l}\preceq\boldsymbol{m}} \zeta_{\mathcal{A}}(\boldsymbol{l}) = (-1)^k \sum_{\boldsymbol{l}} \Biggl(\sum_{\substack{\boldsymbol{m}\succeq\boldsymbol{k}^* \\ \boldsymbol{m}\succeq\boldsymbol{l}}} (-1)^{\operatorname{dep}\boldsymbol{m}} \Biggr) \zeta_{\mathcal{A}}(\boldsymbol{l}).
\end{aligned}$$

Here for indices $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ of weight k, writing $\boldsymbol{k}_1 \vee \boldsymbol{k}_2$ for their coarsest common refinement, we have

$$\sum_{\substack{\boldsymbol{m}\succeq\boldsymbol{k}_1 \\ \boldsymbol{m}\succeq\boldsymbol{k}_2}} (-1)^{\operatorname{dep}\boldsymbol{m}} = \sum_{\boldsymbol{m}\succeq\boldsymbol{k}_1\vee\boldsymbol{k}_2} (-1)^{\operatorname{dep}\boldsymbol{m}} = \begin{cases} (-1)^k & \text{if } \boldsymbol{k}_1 \vee \boldsymbol{k}_2 = (1,\dots,1); \\ 0 & \text{otherwise.} \end{cases}$$

Since $\boldsymbol{k}^* \vee \boldsymbol{l} = (1,\dots,1)$ if and only if $\boldsymbol{l} \succeq \boldsymbol{k}$, we obtain

$$(-1)^n\zeta_{\mathcal{A}}(\boldsymbol{k}) = (-1)^k \sum_{\boldsymbol{l}\succeq\boldsymbol{k}} (-1)^k\zeta_{\mathcal{A}}(\boldsymbol{l}) = \sum_{\boldsymbol{l}\succeq\boldsymbol{k}} \zeta_{\mathcal{A}}(\boldsymbol{l}),$$

as desired. □

Corollary 2.16. *Let $\boldsymbol{k}$ be an index satisfying $\operatorname{dep}\boldsymbol{k} = \operatorname{wt}\boldsymbol{k} - 1$. Such an index can always be written as $\boldsymbol{k} = (\underbrace{1,\dots,1}_{i-1},2,\underbrace{1,\dots,1}_{k-i-1})$ where $1 \leq i \leq k-1$, and we have*

$$\zeta_{\mathcal{A}}(\boldsymbol{k}) = \zeta_{\mathcal{A}}^{\star}(\boldsymbol{k}) = \frac{(-1)^{i+1}}{k}\binom{k}{i}B_{p-k}.$$

In particular, if $\boldsymbol{k}$ has even weight, then $\zeta_{\mathcal{A}}(\boldsymbol{k}) = \zeta_{\mathcal{A}}^{\star}(\boldsymbol{k}) = 0$.

Proof. If we write $\boldsymbol{k} = (k_1, \dots, k_n)$, then Proposition 2.9 gives

$$\zeta_{\mathcal{A}}(\boldsymbol{k}) + (-1)^n \zeta^\star_{\mathcal{A}}(k_n, \dots, k_1) = 0$$

because for $j = 1, \dots, n-1$ either $(k_j, \dots, k_1)$ or $(k_{j+1}, \dots, k_n)$ is an all-one sequence. It follows that

$$0 = \zeta_{\mathcal{A}}(\boldsymbol{k}) + (-1)^n \zeta^\star_{\mathcal{A}}(k_n, \dots, k_1) = \zeta_{\mathcal{A}}(\boldsymbol{k}) + (-1)^{k+n} \zeta^\star_{\mathcal{A}}(\boldsymbol{k}) = \zeta_{\mathcal{A}}(\boldsymbol{k}) - \zeta^\star_{\mathcal{A}}(\boldsymbol{k}).$$

Now Theorem 2.14 and Proposition 2.3 give

$$\zeta^\star_{\mathcal{A}}(\boldsymbol{k}) = -\zeta^\star_{\mathcal{A}}(i, k-i) = \frac{(-1)^{i+1}}{k} \binom{k}{i} B_{p-k}$$

from which the result follows. □

§ 2.7. Result of Pilehrood, Pilehrood, and Tauraso

Pilehrood, Pilehrood, and Tauraso [4] proved relations among FMZ(S)Vs of weight 7 and 9 that cannot be deduced from the relations described so far.

Theorem 2.17 ([4, Corollary 4.3]). *We have*

$$\zeta^\star_{\mathcal{A}}(2,2,2,1) = \frac{27}{16} B_{p-7}, \qquad \zeta^\star_{\mathcal{A}}(6,1,1,1) = \frac{1}{54} B_{p-3}^3 + \frac{1899}{648} B_{p-9}.$$

§ 3. Numerical tables

§ 3.1. Weight 0

The only index of weight 0 is the empty index $\emptyset$. Since $\zeta_{\mathcal{A}}(\emptyset) = \zeta^\star_{\mathcal{A}}(\emptyset) = 1$ by definition, we have $\dim \mathcal{Z}_{\mathcal{A},0} = 1$.

§ 3.2. Weight 1

The only index of weight 1 is (1). Since $\zeta_{\mathcal{A}}(1) = \zeta^\star_{\mathcal{A}}(1) = 0$ by Proposition 2.2, we have $\dim \mathcal{Z}_{\mathcal{A},1} = 0$.

§ 3.3. Weight 2

The indices of weight 2 are (2) and $(1,1)$. Since $\zeta_{\mathcal{A}}(2) = \zeta^\star_{\mathcal{A}}(2) = 0$ by Proposition 2.2 and $\zeta_{\mathcal{A}}(1,1) = \zeta^\star_{\mathcal{A}}(1,1) = 0$ by Corollary 2.4 or Corollary 2.8, we have $\dim \mathcal{Z}_{\mathcal{A},2} = 0$.

§ 3.4. Weight 3

The indices of weight 3 are (3), $(2,1)$, $(1,2)$, and $(1,1,1)$. We have $\zeta_{\mathcal{A}}(3)=\zeta^{\star}_{\mathcal{A}}(3)=0$ by Proposition 2.2 and $\zeta_{\mathcal{A}}(1,1,1)=\zeta^{\star}_{\mathcal{A}}(1,1,1)=0$ by Corollary 2.8; we also have $\zeta_{\mathcal{A}}(2,1)=\zeta^{\star}_{\mathcal{A}}(2,1)=B_{p-3}$ and $\zeta_{\mathcal{A}}(1,2)=\zeta^{\star}_{\mathcal{A}}(1,2)=-B_{p-3}$ by Proposition 2.3. It follows that $\dim\mathcal{Z}_{\mathcal{A},3}\le 1$; it appears open whether $\dim\mathcal{Z}_{\mathcal{A},3}=1$, i.e. whether $B_{p-3}\neq 0$ for infinitely many primes p.

§ 3.5. Weight 4

Let $\boldsymbol{k}$ be an index of weight 4. If $\boldsymbol{k}$ has depth 1 or 4, then $\zeta_{\mathcal{A}}(\boldsymbol{k})=\zeta^{\star}_{\mathcal{A}}(\boldsymbol{k})=0$. If $\boldsymbol{k}$ has depth 2, then $\zeta_{\mathcal{A}}(\boldsymbol{k})=\zeta^{\star}_{\mathcal{A}}(\boldsymbol{k})=0$ by Corollary 2.4. If $\boldsymbol{k}$ has depth 3, then $\zeta^{\star}_{\mathcal{A}}(\boldsymbol{k})=-\zeta^{\star}_{\mathcal{A}}(\boldsymbol{k}^{*})=0$ by Theorem 2.14 and by the fact that $\boldsymbol{k}^{*}$ has depth 2. It follows that $\dim\mathcal{Z}_{\mathcal{A},4}=0$.

§ 3.6. Weight 5

Proposition 3.1. *Let $\boldsymbol{k}$ be an index of weight 5. Then the values of $\zeta_{\mathcal{A}}(\boldsymbol{k})$ and $\zeta^{\star}_{\mathcal{A}}(\boldsymbol{k})$ are as given in Table 2 in terms of B_{p-5}.*

Proof. If $\operatorname{dep}\boldsymbol{k}=2$, use Proposition 2.3. If $\operatorname{dep}\boldsymbol{k}=3$, use Corollary 2.10. If $\operatorname{dep}\boldsymbol{k}=4$, use Corollary 2.16. □

Table 2. FMZ(S)Vs of weight 5 in terms of B_{p-5}; e.g. $\zeta_{\mathcal{A}}(4,1)=\zeta^{\star}_{\mathcal{A}}(4,1)=B_{p-5}$, $\zeta_{\mathcal{A}}(3,1,1)=B_{p-5}/2$, and $\zeta^{\star}_{\mathcal{A}}(3,1,1)=-B_{p-5}/2$. If $\operatorname{wt}\boldsymbol{k}=5$ and $\operatorname{dep}\boldsymbol{k}\in\{1,5\}$, then $\zeta_{\mathcal{A}}(\boldsymbol{k})=\zeta^{\star}_{\mathcal{A}}(\boldsymbol{k})=0$.

$\boldsymbol{k}$	$\zeta_{\mathcal{A}}=\zeta^{\star}_{\mathcal{A}}$	$\boldsymbol{k}$	$\zeta_{\mathcal{A}}$	$\zeta^{\star}_{\mathcal{A}}$	$\boldsymbol{k}$	$\zeta_{\mathcal{A}}=\zeta^{\star}_{\mathcal{A}}$
$(4,1)$	1	$(3,1,1)$	$1/2$	$-1/2$	$(2,1,1,1)$	1
$(3,2)$	-2	$(2,2,1)$	$-3/2$	$3/2$	$(1,2,1,1)$	-2
$(2,3)$	2	$(2,1,2)$	0	0	$(1,1,2,1)$	2
$(1,4)$	-1	$(1,3,1)$	0	0	$(1,1,1,2)$	-1
		$(1,2,2)$	$3/2$	$-3/2$		
		$(1,1,3)$	$-1/2$	$1/2$		

§ 3.7. Weight 6

Proposition 3.2. *Let $\boldsymbol{k}$ be an index of weight 6. Then the values of $\zeta_{\mathcal{A}}(\boldsymbol{k})$ and $\zeta^\star_{\mathcal{A}}(\boldsymbol{k})$ are as given in Table 3 in terms of B^2_{p-3}.*

Proof. If $\operatorname{dep}\boldsymbol{k} \in \{1,2,5,6\}$, then $\zeta_{\mathcal{A}}(\boldsymbol{k}) = \zeta^\star_{\mathcal{A}}(\boldsymbol{k}) = 0$ (use Corollary 2.16 when $\operatorname{dep}\boldsymbol{k} = 5$).

Note that if $\operatorname{dep}\boldsymbol{k} = 3$, then $\zeta_{\mathcal{A}}(\boldsymbol{k}) = \zeta^\star_{\mathcal{A}}(\boldsymbol{k})$ by Corollary 2.5.

We have $\zeta_{\mathcal{A}}(2,2,2) = \zeta^\star_{\mathcal{A}}(2,2,2) = 0$ by Corollary 2.8, and so $\zeta^\star_{\mathcal{A}}(1,2,2,1) = -\zeta^\star_{\mathcal{A}}(2,2,2) = 0$ by Theorem 2.14. It follows that

$$\zeta_{\mathcal{A}}(1,2,2,1) = -\zeta^\star_{\mathcal{A}}(1,2,2,1) + \zeta_{\mathcal{A}}(1,2)\zeta_{\mathcal{A}}(2,1) = -0 + (-B_{p-3})B_{p-3} = -B^2_{p-3}$$

by Corollary 2.11.

Set $a = \zeta_{\mathcal{A}}(4,1,1) = \zeta^\star_{\mathcal{A}}(4,1,1)$. Then by Proposition 2.6, Corollary 2.11, and Theorem 2.14, we have $\zeta_{\mathcal{A}}(1,1,4) = \zeta^\star_{\mathcal{A}}(1,1,4) = \zeta_{\mathcal{A}}(3,1,1,1) = \zeta_{\mathcal{A}}(1,1,1,3) = a$ and $\zeta^\star_{\mathcal{A}}(3,1,1,1) = \zeta^\star_{\mathcal{A}}(1,1,1,3) = -a$. Therefore Propositions 2.6 and 2.7 show that $\zeta_{\mathcal{A}}(1,4,1) = \zeta^\star_{\mathcal{A}}(1,4,1) = -2a$, $\zeta_{\mathcal{A}}(1,1,3,1) = \zeta_{\mathcal{A}}(1,3,1,1) = -a$, and $\zeta^\star_{\mathcal{A}}(1,1,3,1) = \zeta^\star_{\mathcal{A}}(1,3,1,1) = a$. Corollaries 2.15, 2.16, and 2.8 show that

$$-\zeta_{\mathcal{A}}(1,4,1) = \zeta_{\mathcal{A}}(1,4,1) + \zeta_{\mathcal{A}}(1,3,1,1) + \zeta_{\mathcal{A}}(1,2,2,1) + \zeta_{\mathcal{A}}(1,1,3,1),$$

from which it follows that $2a = -2a - a - B^2_{p-3} - a$, i.e. $a = -B^2_{p-3}/6$. Then Theorem 2.14 gives $\zeta^\star(3,1,2)$, $\zeta^\star(2,1,3)$, and $\zeta^\star_{\mathcal{A}}(2,1,1,2)$. Use Corollary 2.11 to obtain $\zeta_{\mathcal{A}}(2,1,1,2)$.

Table 3. FMZ(S)Vs of weight 6 in terms of B^2_{p-3}; e.g. $\zeta_{\mathcal{A}}(4,1,1) = \zeta^\star_{\mathcal{A}}(4,1,1) = -B^2_{p-3}/6$, $\zeta_{\mathcal{A}}(3,1,1,1) = -B^2_{p-3}/6$, and $\zeta^\star_{\mathcal{A}}(3,1,1,1) = B^2_{p-3}/6$. If $\operatorname{wt}\boldsymbol{k} = 6$ and $\operatorname{dep}\boldsymbol{k} \in \{1,2,5,6\}$, then $\zeta_{\mathcal{A}}(\boldsymbol{k}) = \zeta^\star_{\mathcal{A}}(\boldsymbol{k}) = 0$.

$\boldsymbol{k}$	$\zeta_{\mathcal{A}} = \zeta^\star_{\mathcal{A}}$	$\boldsymbol{k}$	$\zeta_{\mathcal{A}}$	$\zeta^\star_{\mathcal{A}}$
$(4,1,1)$	$-1/6$	$(3,1,1,1)$	$-1/6$	$1/6$
$(3,2,1)$	$2/6$	$(2,2,1,1)$	$2/6$	$-2/6$
$(3,1,2)$	$1/6$	$(2,1,2,1)$	$3/6$	$3/6$
$(2,3,1)$	$-3/6$	$(2,1,1,2)$	$-4/6$	$-2/6$
$(2,2,2)$	$0/6$	$(1,3,1,1)$	$1/6$	$-1/6$
$(2,1,3)$	$1/6$	$(1,2,2,1)$	$-6/6$	$0/6$
$(1,4,1)$	$2/6$	$(1,2,1,2)$	$3/6$	$3/6$
$(1,3,2)$	$-3/6$	$(1,1,3,1)$	$1/6$	$-1/6$
$(1,2,3)$	$2/6$	$(1,1,2,2)$	$2/6$	$-2/6$
$(1,1,4)$	$-1/6$	$(1,1,1,3)$	$-1/6$	$1/6$

In a similar manner, if we set $b = \zeta_{\mathcal{A}}(3,2,1) = \zeta^\star_{\mathcal{A}}(3,2,1)$, then by Proposition 2.6, Proposition 2.7, Corollary 2.11, and Theorem 2.14 we have $\zeta_{\mathcal{A}}(1,2,3) = \zeta^\star_{\mathcal{A}}(1,2,3) = b$, $\zeta_{\mathcal{A}}(1,3,2) = \zeta^\star_{\mathcal{A}}(1,3,2) = \zeta_{\mathcal{A}}(2,3,1) = \zeta^\star_{\mathcal{A}}(2,3,1) = -b - B_{p-3}^2/6$, $\zeta^\star_{\mathcal{A}}(2,2,1,1) = \zeta^\star_{\mathcal{A}}(1,1,2,2) = -b$, $\zeta_{\mathcal{A}}(2,2,1,1) = \zeta_{\mathcal{A}}(1,1,2,2) = b$, $\zeta^\star_{\mathcal{A}}(2,1,2,1) = \zeta^\star_{\mathcal{A}}(1,2,1,2) = b + B_{p-3}^2/6$, and $\zeta_{\mathcal{A}}(2,1,2,1) = \zeta_{\mathcal{A}}(1,2,1,2) = -b + 5B_{p-3}^2/6$. Corollary 2.15 shows that

$$-\zeta_{\mathcal{A}}(3,2,1) = \zeta_{\mathcal{A}}(3,2,1) + \zeta_{\mathcal{A}}(2,1,2,1) + \zeta_{\mathcal{A}}(1,2,2,1) + \zeta_{\mathcal{A}}(3,1,1,1),$$

from which it follows that $-b = b + (-b + 5B_{p-3}^2/6) - B_{p-3}^2 - B_{p-3}^2/6$, i.e. $b = B_{p-3}^2/3$. $\square$

§ 3.8. Weight 7

Theorem 3.3. *Let $\boldsymbol{k}$ be an index of weight 7. Then the values of $\zeta_{\mathcal{A}}(\boldsymbol{k})$ and $\zeta^\star_{\mathcal{A}}(\boldsymbol{k})$ are as given in Table 4 in terms of B_{p-7}.*

Proof. If $\operatorname{dep}\boldsymbol{k} \in \{1,2,3,6,7\}$, then it is easy to compute $\zeta_{\mathcal{A}}(\boldsymbol{k})$ and $\zeta^\star_{\mathcal{A}}(\boldsymbol{k})$.

Apply Proposition 2.9 to $\boldsymbol{k}$. If we write $\boldsymbol{k}$ as a concatenation of two nonempty indices, then one of the two has weight in $\{1,2,4\}$ and so its $\zeta_{\mathcal{A}}$ and $\zeta^\star_{\mathcal{A}}$ are zero. It follows that $\zeta_{\mathcal{A}}(\boldsymbol{k}) + (-1)^{\operatorname{dep}\boldsymbol{k}}\zeta^\star_{\mathcal{A}}(\boldsymbol{k}') = 0$, where $\boldsymbol{k}'$ is the reversal of $\boldsymbol{k}$, and therefore we have $\zeta_{\mathcal{A}}(\boldsymbol{k}) = (-1)^{\operatorname{dep}\boldsymbol{k}}\zeta^\star_{\mathcal{A}}(\boldsymbol{k})$ by Proposition 2.6 because $\operatorname{wt}\boldsymbol{k} = 7$ is odd.

Suppose that $\boldsymbol{k}$ has depth 5. Theorem 2.14 allows us to compute $\zeta^\star_{\mathcal{A}}(\boldsymbol{k})$ because $\boldsymbol{k}^*$ has depth 3 and so we already know the value of $\zeta^\star_{\mathcal{A}}(\boldsymbol{k}^*)$. We can then use $\zeta_{\mathcal{A}}(\boldsymbol{k}) = -\zeta^\star_{\mathcal{A}}(\boldsymbol{k})$ deduced above.

In what follows we shall compute $\zeta_{\mathcal{A}}(\boldsymbol{k})$ and $\zeta^\star_{\mathcal{A}}(\boldsymbol{k})$ for indices of depth 4. Note that $\zeta_{\mathcal{A}}(\boldsymbol{k}) = \zeta^\star_{\mathcal{A}}(\boldsymbol{k})$ for all such indices $\boldsymbol{k}$. Since Theorem 2.17 gives the value of $\zeta_{\mathcal{A}}(2,2,2,1)$, it suffices to express the other FMZ(S)Vs in terms of $\zeta_{\mathcal{A}}(2,2,2,1)$ and B_{p-7}.

Set $\alpha = \zeta(4,1,1,1)$. Applying the harmonic product rule to $\zeta_{\mathcal{A}}(4,1,1)\zeta_{\mathcal{A}}(1)$, we obtain $\zeta_{\mathcal{A}}(1,4,1,1) = -3\alpha + 3B_{p-7}$. Then Proposition 2.6 allows us to find $\zeta_{\mathcal{A}}(1,1,4,1) = 3\alpha - 3B_{p-7}$ and $\zeta_{\mathcal{A}}(1,1,1,4) = -\alpha$. Thus Theorem 2.14 gives $\zeta_{\mathcal{A}}(3,1,1,2) = -3\alpha + 3B_{p-7}$ and $\zeta_{\mathcal{A}}(2,1,1,3) = 3\alpha - 3B_{p-7}$.

Set $\beta = \zeta_{\mathcal{A}}(2,2,2,1)$. We may similarly obtain $\zeta_{\mathcal{A}}(2,2,1,2) = -3\beta + 9B_{p-7}$, $\zeta_{\mathcal{A}}(2,1,2,2) = 3\beta - 9B_{p-7}$, $\zeta_{\mathcal{A}}(1,2,2,2) = -\beta$, $\zeta_{\mathcal{A}}(1,3,2,1) = -3\beta + 9B_{p-7}$, and $\zeta_{\mathcal{A}}(1,2,3,1) = 3\beta - 9B_{p-7}$.

Set $\gamma = \zeta_{\mathcal{A}}(3,2,1,1)$, $\delta = \zeta_{\mathcal{A}}(3,1,2,1)$, and $\varepsilon = \zeta_{\mathcal{A}}(2,1,3,1)$. Then Proposition 2.6 and Theorem 2.14 give all the remaining values. We look at $\zeta_{\mathcal{A}}(3,2,1)\zeta_{\mathcal{A}}(1)$, $\zeta_{\mathcal{A}}(3,1,2)\zeta_{\mathcal{A}}(1)$, and $\zeta_{\mathcal{A}}(2,3,1)\zeta_{\mathcal{A}}(1)$ to obtain $\gamma = -\alpha + 2\beta - 6B_{p-7}$, $\delta = 2\alpha - \beta$, and $\varepsilon = -4\alpha - \beta + 9B_{p-7}$. We then express $\zeta^\star_{\mathcal{A}}(2,2,1,1,1)$ as a sum of FMZVs to get $\alpha = \beta$. $\square$

Table 4. FMZ(S)Vs of weight 7 in terms of B_{p-7}; e.g. $\zeta_{\mathcal{A}}(6,1) = \zeta^\star_{\mathcal{A}}(6,1) = B_{p-7}$, $\zeta_{\mathcal{A}}(5,1,1) = B_{p-7}$, and $\zeta^\star_{\mathcal{A}}(5,1,1) = -B_{p-7}$. If $\operatorname{wt}\boldsymbol{k} = 7$ and $\operatorname{dep}\boldsymbol{k} \in \{1,7\}$, then $\zeta_{\mathcal{A}}(\boldsymbol{k}) = \zeta^\star_{\mathcal{A}}(\boldsymbol{k}) = 0$.

$\boldsymbol{k}$	$\zeta_{\mathcal{A}} = \zeta^\star_{\mathcal{A}}$	$\boldsymbol{k}$	$\zeta_{\mathcal{A}}$	$\zeta^\star_{\mathcal{A}}$	$\boldsymbol{k}$	$\zeta_{\mathcal{A}}$	$\zeta^\star_{\mathcal{A}}$
$(6,1)$	1	$(5,1,1)$	1	-1	$(3,1,1,1,1)$	1	-1
$(5,2)$	-3	$(4,2,1)$	-3	3	$(2,2,1,1,1)$	-3	3
$(4,3)$	5	$(4,1,2)$	-1	1	$(2,1,2,1,1)$	2	-2
$(3,4)$	-5	$(3,3,1)$	2	-2	$(2,1,1,2,1)$	-2	2
$(2,5)$	3	$(3,2,2)$	4	-4	$(2,1,1,1,2)$	0	0
$(1,6)$	-1	$(3,1,3)$	0	0	$(1,3,1,1,1)$	-1	1
		$(2,4,1)$	-2	2	$(1,2,2,1,1)$	4	-4
		$(2,3,2)$	0	0	$(1,2,1,2,1)$	0	0
		$(2,2,3)$	-4	4	$(1,2,1,1,2)$	2	-2
		$(2,1,4)$	1	-1	$(1,1,3,1,1)$	0	0
		$(1,5,1)$	0	0	$(1,1,2,2,1)$	-4	4
		$(1,4,2)$	2	-2	$(1,1,2,1,2)$	-2	2
		$(1,3,3)$	-2	2	$(1,1,1,3,1)$	1	-1
		$(1,2,4)$	3	-3	$(1,1,1,2,2)$	3	-3
		$(1,1,5)$	-1	1	$(1,1,1,1,3)$	-1	1

$\boldsymbol{k}$	$\zeta_{\mathcal{A}} = \zeta^\star_{\mathcal{A}}$	$\boldsymbol{k}$	$\zeta_{\mathcal{A}} = \zeta^\star_{\mathcal{A}}$	$\boldsymbol{k}$	$\zeta_{\mathcal{A}} = \zeta^\star_{\mathcal{A}}$
$(4,1,1,1)$	$27/16$	$(1,4,1,1)$	$-33/16$	$(2,1,1,1,1,1)$	1
$(3,2,1,1)$	$-69/16$	$(1,3,2,1)$	$63/16$	$(1,2,1,1,1,1)$	-3
$(3,1,2,1)$	$27/16$	$(1,3,1,2)$	$-\ 9/16$	$(1,1,2,1,1,1)$	5
$(3,1,1,2)$	$-33/16$	$(1,2,3,1)$	$-63/16$	$(1,1,1,2,1,1)$	-5
$(2,3,1,1)$	$27/16$	$(1,2,2,2)$	$-27/16$	$(1,1,1,1,2,1)$	3
$(2,2,2,1)$	$27/16$	$(1,2,1,3)$	$-27/16$	$(1,1,1,1,1,2)$	-1
$(2,2,1,2)$	$63/16$	$(1,1,4,1)$	$33/16$		
$(2,1,3,1)$	$9/16$	$(1,1,3,2)$	$-27/16$		
$(2,1,2,2)$	$-63/16$	$(1,1,2,3)$	$69/16$		
$(2,1,1,3)$	$33/16$	$(1,1,1,4)$	$-27/16$		

§ 3.9. Weight 8

Theorem 3.4. *Let $\boldsymbol{k}$ be an index of weight 8. Then the values of $\zeta_{\mathcal{A}}(\boldsymbol{k})$ and $\zeta^{\star}_{\mathcal{A}}(\boldsymbol{k})$ are as given in Tables 5–8 in terms of $\zeta_{\mathcal{A}}(6,1,1)$ and $B_{p-3}B_{p-5}$.*

Proof. If $\operatorname{dep}\boldsymbol{k} \in \{1,2,7,8\}$, then $\zeta_{\mathcal{A}}(\boldsymbol{k}) = \zeta^{\star}_{\mathcal{A}}(\boldsymbol{k}) = 0$. See [1, Theorem 7.4] for how to get the values $\zeta^{\star}_{\mathcal{A}}(\boldsymbol{k})$ when $\operatorname{dep}\boldsymbol{k} = 3$. If $\operatorname{dep}\boldsymbol{k} = 3$, then Corollary 2.5 gives the value of $\zeta_{\mathcal{A}}(\boldsymbol{k})$. If $\operatorname{dep}\boldsymbol{k} = 4$, then since we can compute the value of $\zeta^{\star}_{\mathcal{A}}(\boldsymbol{k}) - \zeta_{\mathcal{A}}(\boldsymbol{k})$ and Corollary 2.11 gives the value of $\zeta_{\mathcal{A}}(\boldsymbol{k}) + \zeta^{\star}_{\mathcal{A}}(\boldsymbol{k})$, we can get the values of $\zeta_{\mathcal{A}}(\boldsymbol{k})$ and $\zeta^{\star}_{\mathcal{A}}(\boldsymbol{k})$. If $\operatorname{dep}\boldsymbol{k} \in \{5,6\}$, then Theorem 2.14 gives the value of $\zeta^{\star}_{\mathcal{A}}(\boldsymbol{k})$ and we can get the value of $\zeta_{\mathcal{A}}(\boldsymbol{k})$ by using Proposition 2.9. □

Table 5. FMZ(S)Vs of weight 8 and depth 3 in terms of $\zeta_{\mathcal{A}}(6,1,1)$ and $B_{p-3}B_{p-5}$; e.g. $\zeta_{\mathcal{A}}(5,2,1) = \zeta^{\star}_{\mathcal{A}}(5,2,1) = \zeta_{\mathcal{A}}(6,1,1) + B_{p-3}B_{p-5}$.

$\boldsymbol{k}$	$\zeta_{\mathcal{A}}(6,1,1)$-component of $\zeta_{\mathcal{A}} = \zeta^{\star}_{\mathcal{A}}$	$B_{p-3}B_{p-5}$-component of $\zeta_{\mathcal{A}} = \zeta^{\star}_{\mathcal{A}}$	$\boldsymbol{k}$	$\zeta_{\mathcal{A}}(6,1,1)$-component of $\zeta_{\mathcal{A}} = \zeta^{\star}_{\mathcal{A}}$	$B_{p-3}B_{p-5}$-component of $\zeta_{\mathcal{A}} = \zeta^{\star}_{\mathcal{A}}$
$(6,1,1)$	$4/4$	0	$(2,4,2)$	$-80/4$	$-16/4$
$(5,2,1)$	$4/4$	$4/4$	$(2,3,3)$	$40/4$	$8/4$
$(5,1,2)$	$-14/4$	$-2/4$	$(2,2,4)$	$40/4$	$8/4$
$(4,3,1)$	$-50/4$	$-18/4$	$(2,1,5)$	$-14/4$	$-2/4$
$(4,2,2)$	$40/4$	$8/4$	$(1,6,1)$	$-8/4$	0
$(4,1,3)$	$10/4$	$2/4$	$(1,5,2)$	$10/4$	$-2/4$
$(3,4,1)$	$40/4$	$16/4$	$(1,4,3)$	$40/4$	$16/4$
$(3,3,2)$	$40/4$	$8/4$	$(1,3,4)$	$-50/4$	$-18/4$
$(3,2,3)$	$-80/4$	$-16/4$	$(1,2,5)$	$4/4$	$4/4$
$(3,1,4)$	$10/4$	$2/4$	$(1,1,6)$	$4/4$	0
$(2,5,1)$	$10/4$	$-2/4$			

Table 6. FMZ(S)Vs of weight 8 and depth 4 in terms of $\zeta_{\mathcal{A}}(6,1,1)$ and $B_{p-3}B_{p-5}$; e.g. $\zeta_{\mathcal{A}}(5,1,1,1) = 3\zeta_{\mathcal{A}}(6,1,1)/4 - B_{p-3}B_{p-5}/4$ and $\zeta^{\star}_{\mathcal{A}}(5,1,1,1) = -3\zeta_{\mathcal{A}}(6,1,1)/4 + B_{p-3}B_{p-5}/4$.

$\boldsymbol{k}$	$\zeta_{\mathcal{A}}(6,1,1)$-component of $\zeta_{\mathcal{A}}$	$B_{p-3}B_{p-5}$-component of $\zeta_{\mathcal{A}}$	$\zeta_{\mathcal{A}}(6,1,1)$-component of $\zeta^{\star}_{\mathcal{A}}$	$B_{p-3}B_{p-5}$-component of $\zeta^{\star}_{\mathcal{A}}$
$(5,1,1,1)$	$3/4$	$-1/4$	$-3/4$	$1/4$
$(4,2,1,1)$	$3/4$	$5/4$	$-3/4$	$-5/4$
$(4,1,2,1)$	$18/4$	$8/4$	$-18/4$	$-4/4$
$(4,1,1,2)$	$-18/4$	$-6/4$	$18/4$	$2/4$
$(3,3,1,1)$	$-42/4$	$-12/4$	$42/4$	$12/4$
$(3,2,2,1)$	$18/4$	$-6/4$	$-18/4$	$-2/4$
$(3,2,1,2)$	$27/4$	$9/4$	$-27/4$	$-1/4$
$(3,1,3,1)$	0	0	0	0
$(3,1,2,2)$	$-45/4$	$-9/4$	$45/4$	$9/4$
$(3,1,1,3)$	$30/4$	$6/4$	$-30/4$	$-6/4$
$(2,4,1,1)$	$33/4$	$9/4$	$-33/4$	$-9/4$
$(2,3,2,1)$	$-27/4$	$-1/4$	$27/4$	$9/4$
$(2,3,1,2)$	$27/4$	$1/4$	$-27/4$	$-9/4$
$(2,2,3,1)$	0	$6/4$	0	$-6/4$
$(2,2,2,2)$	0	0	0	0
$(2,2,1,3)$	$-45/4$	$-9/4$	$45/4$	$9/4$
$(2,1,4,1)$	$-18/4$	$-4/4$	$18/4$	$8/4$
$(2,1,3,2)$	$27/4$	$1/4$	$-27/4$	$-9/4$
$(2,1,2,3)$	$27/4$	$9/4$	$-27/4$	$-1/4$
$(2,1,1,4)$	$-18/4$	$-6/4$	$18/4$	$2/4$
$(1,5,1,1)$	$-3/4$	$1/4$	$3/4$	$-1/4$
$(1,4,2,1)$	$-18/4$	$-12/4$	$18/4$	$8/4$
$(1,4,1,2)$	$-18/4$	$-4/4$	$18/4$	$8/4$
$(1,3,3,1)$	$54/4$	$18/4$	$-54/4$	$-18/4$
$(1,3,2,2)$	0	$6/4$	0	$-6/4$
$(1,3,1,3)$	0	0	0	0
$(1,2,4,1)$	$-18/4$	$-12/4$	$18/4$	$8/4$
$(1,2,3,2)$	$-27/4$	$-1/4$	$27/4$	$9/4$
$(1,2,2,3)$	$18/4$	$-6/4$	$-18/4$	$-2/4$
$(1,2,1,4)$	$18/4$	$8/4$	$-18/4$	$-4/4$
$(1,1,5,1)$	$-3/4$	$1/4$	$3/4$	$-1/4$
$(1,1,4,2)$	$33/4$	$9/4$	$-33/4$	$-9/4$
$(1,1,3,3)$	$-42/4$	$-12/4$	$42/4$	$12/4$
$(1,1,2,4)$	$3/4$	$5/4$	$-3/4$	$-5/4$
$(1,1,1,5)$	$3/4$	$-1/4$	$-3/4$	$1/4$

Table 7. FMZ(S)Vs of weight 8 and depth 5 in terms of $\zeta_{\mathcal{A}}(6,1,1)$ and $B_{p-3}B_{p-5}$; e.g. $\zeta_{\mathcal{A}}(4,1,1,1,1) = \zeta^{\star}_{\mathcal{A}}(4,1,1,1,1) = 3\zeta_{\mathcal{A}}(6,1,1)/4 - B_{p-3}B_{p-5}/4$.

$\boldsymbol{k}$	$\zeta_{\mathcal{A}}(6,1,1)$-component of $\zeta_{\mathcal{A}}$	$B_{p-3}B_{p-5}$-component of $\zeta_{\mathcal{A}}$	$\zeta_{\mathcal{A}}(6,1,1)$-component of $\zeta^{\star}_{\mathcal{A}}$	$B_{p-3}B_{p-5}$-component of $\zeta^{\star}_{\mathcal{A}}$
$(4,1,1,1,1)$	$3/4$	$-1/4$	$3/4$	$-1/4$
$(3,2,1,1,1)$	$3/4$	$5/4$	$3/4$	$5/4$
$(3,1,2,1,1)$	$-42/4$	$-12/4$	$-42/4$	$-12/4$
$(3,1,1,2,1)$	$33/4$	$11/4$	$33/4$	$9/4$
$(3,1,1,1,2)$	$-3/4$	$-1/4$	$-3/4$	$1/4$
$(2,3,1,1,1)$	$18/4$	$4/4$	$18/4$	$4/4$
$(2,2,2,1,1)$	$18/4$	$2/4$	$18/4$	$2/4$
$(2,2,1,2,1)$	$-27/4$	$-15/4$	$-27/4$	$-9/4$
$(2,2,1,1,2)$	$-18/4$	$-2/4$	$-18/4$	$-8/4$
$(2,1,3,1,1)$	0	$2/4$	0	0
$(2,1,2,2,1)$	0	0	0	$6/4$
$(2,1,2,1,2)$	$54/4$	$18/4$	$54/4$	$18/4$
$(2,1,1,3,1)$	$-18/4$	$-8/4$	$-18/4$	$-8/4$
$(2,1,1,2,2)$	$-18/4$	$-2/4$	$-18/4$	$-8/4$
$(2,1,1,1,3)$	$-3/4$	$-1/4$	$-3/4$	$1/4$
$(1,4,1,1,1)$	$-18/4$	$-2/4$	$-18/4$	$-2/4$
$(1,3,2,1,1)$	$27/4$	$1/4$	$27/4$	$1/4$
$(1,3,1,2,1)$	$27/4$	$9/4$	$27/4$	$9/4$
$(1,3,1,1,2)$	$-18/4$	$-8/4$	$-18/4$	$-8/4$
$(1,2,3,1,1)$	$-45/4$	$-11/4$	$-45/4$	$-9/4$
$(1,2,2,2,1)$	0	$12/4$	0	0
$(1,2,2,1,2)$	0	0	0	$6/4$
$(1,2,1,3,1)$	$27/4$	$9/4$	$27/4$	$9/4$
$(1,2,1,2,2)$	$-27/4$	$-15/4$	$-27/4$	$-9/4$
$(1,2,1,1,3)$	$33/4$	$11/4$	$33/4$	$9/4$
$(1,1,4,1,1)$	$30/4$	$6/4$	$30/4$	$6/4$
$(1,1,3,2,1)$	$-45/4$	$-11/4$	$-45/4$	$-9/4$
$(1,1,3,1,2)$	0	$2/4$	0	0
$(1,1,2,3,1)$	$27/4$	$1/4$	$27/4$	$1/4$
$(1,1,2,2,2)$	$18/4$	$2/4$	$18/4$	$2/4$
$(1,1,2,1,3)$	$-42/4$	$-12/4$	$-42/4$	$-12/4$
$(1,1,1,4,1)$	$-18/4$	$-2/4$	$-18/4$	$-2/4$
$(1,1,1,3,2)$	$18/4$	$4/4$	$18/4$	$4/4$
$(1,1,1,2,3)$	$3/4$	$5/4$	$3/4$	$5/4$
$(1,1,1,1,4)$	$3/4$	$-1/4$	$3/4$	$-1/4$

Table 8. FMZ(S)Vs of weight 8 and or 6 in terms of $\zeta_{\mathcal{A}}(6,1,1)$ and $B_{p-3}B_{p-5}$; e.g. $\zeta_{\mathcal{A}}(2,2,1,1,1,1) = \zeta_{\mathcal{A}}(6,1,1) + B_{p-3}B_{p-5}$ and $\zeta^{\star}_{\mathcal{A}}(2,2,1,1,1,1) = -\zeta_{\mathcal{A}}(6,1,1) - B_{p-3}B_{p-5}$.

$\boldsymbol{k}$	$\zeta_{\mathcal{A}}(6,1,1)$-component of $\zeta_{\mathcal{A}}$	$B_{p-3}B_{p-5}$-component of $\zeta_{\mathcal{A}}$	$\zeta_{\mathcal{A}}(6,1,1)$-component of $\zeta^{\star}_{\mathcal{A}}$	$B_{p-3}B_{p-5}$-component of $\zeta^{\star}_{\mathcal{A}}$
$(3,1,1,1,1,1)$	$4/4$	0	$-4/4$	0
$(2,2,1,1,1,1)$	$4/4$	$4/4$	$-4/4$	$-4/4$
$(2,1,2,1,1,1)$	$-50/4$	$-14/4$	$50/4$	$18/4$
$(2,1,1,2,1,1)$	$40/4$	$8/4$	$-40/4$	$-16/4$
$(2,1,1,1,2,1)$	$10/4$	$10/4$	$-10/4$	$2/4$
$(2,1,1,1,1,2)$	$-8/4$	$-8/4$	$8/4$	0
$(1,3,1,1,1,1)$	$-14/4$	$-2/4$	$14/4$	$2/4$
$(1,2,2,1,1,1)$	$40/4$	$4/4$	$-40/4$	$-8/4$
$(1,2,1,2,1,1)$	$40/4$	$16/4$	$-40/4$	$-8/4$
$(1,2,1,1,2,1)$	$-80/4$	$-32/4$	$80/4$	$16/4$
$(1,2,1,1,1,2)$	$10/4$	$10/4$	$-10/4$	$2/4$
$(1,1,3,1,1,1)$	$10/4$	$2/4$	$-10/4$	$-2/4$
$(1,1,2,2,1,1)$	$-80/4$	$-16/4$	$80/4$	$16/4$
$(1,1,2,1,2,1)$	$40/4$	$16/4$	$-40/4$	$-8/4$
$(1,1,2,1,1,2)$	$40/4$	$8/4$	$-40/4$	$-16/4$
$(1,1,1,3,1,1)$	$10/4$	$2/4$	$-10/4$	$-2/4$
$(1,1,1,2,2,1)$	$40/4$	$4/4$	$-40/4$	$-8/4$
$(1,1,1,2,1,2)$	$-50/4$	$-14/4$	$50/4$	$18/4$
$(1,1,1,1,3,1)$	$-14/4$	$-2/4$	$14/4$	$2/4$
$(1,1,1,1,2,2)$	$4/4$	$4/4$	$-4/4$	$-4/4$
$(1,1,1,1,1,3)$	$4/4$	0	$-4/4$	0

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